NHCU-HEP-94-23

Nonlocal Matrix Generalizations of N=2 Super Virasoro Algebra

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Abstract

We study the generalization of the second Gelfand-Dickey bracket to the superdifferential operators with matrix-valued coefficients. The associated matrix Miura transformation is derived. Using this bracket we work out a nonlocal and nonlinear N=2 superalgebra which contains the N=2 super Virasoro algebra as a subalgebra. The bosonic limit of this superalgebra is considered. We show that when the spin-1 fields in this bosonic algebra are set to zero the resulting Dirac bracket gives precisely the recently derived $V_{2,2}$ algebra.

1. Introduction

The connections between Hamiltonian structures of integrable systems and classical extended conformal algebras are now well understood [1-4]. The extensions of conformal algebras by higher spin fields and their supersymmetric versions are known to given by the Gelfand-Dickey brackets associated with appropriated (super) pseudodifferential operators [5-10].

Recently, in the study of simplest 1+1 dimensional non-abelian Toda field theory, an interesting nonlocal extended conformal algebra called V algebra is discovered [11]. This algebra consists of a Virasoro generator t and two additional spin-2 fields $v_\pm$:

$$\begin{aligned}
\{t(x),t(y)\} &= [-\frac{1}{2}\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y) \\
\{v_\pm(x),t(y)\} &= [2v_\pm(x)\partial_x + v'_\pm(x)\delta(x-y) \\
\{v_\pm(x),v_\pm(y)\} &= 2\epsilon(x-y)v_\pm(x)v_\pm(y) \\
\{v_\pm(x),v_\mp(y)\} &= -2\epsilon(x-y)v_\pm(x)v_\mp(y) \\
&\qquad + [-\frac{1}{2}\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y)
\end{aligned} \tag{1}$$

where $\epsilon(x-y) = \partial_x^{-1}\delta(x-y)$ (This differs from the definition used in Ref.[11] by a factor of 2.) It is realized that this algebra is actually given by the second Gelfand-Dickey bracket associated with the second order differential operator with 2×2 matrix-valued coefficients:

$$L = \partial^2 - U \tag{2}$$

where

$$U = \begin{pmatrix} t & -\sqrt{2}v_+ \\ -\sqrt{2}v_- & t \end{pmatrix} \tag{3}$$

In this correspondence, it is understood that the nonlocal terms arise as a consequnce of the noncommutative character of matrix multiplication [11]. This discovery leads to the generalization of the second Gelfand-Dickey to the differential operators with matrix-valued coefficients. The resulting algebras are nonlocal matrix generalization of W algebras [12].

The purpose of this paper is to study the supersymmetric version of the above generalization. More pecisely, we consider the second Gelfand-Dickey bracket associated superdifferential operators with matrix-valued coifficients. The corresponding matrix Muira transformation is presented and used to prove the Jocobi identity for the second Gelfand-Dickey bracket. We work out explicitly the bracket associated with the third order superdifferential operators with 2×2 matrix-valued coefficients. The resulting superalgebra, which contains the classical N=2 Super Virasoro algebra as a subalgebra, is a nonlocal matrix generalization of N=2 super Virasoro algebra. We then consider further the bosonic limit of this superalgebra. In particular, we show explicitly that the $V_{2,2}$ algebra obtained in Ref.[12] arises from this bosonic algebra as the Dirac bracket when the spin-1 fields are all set to zero.

2. Second Gelfand-Dickey Bracket and Its Miura Transformation

The needed ingredients to generalize the second Gelfand-Dickey bracket to the case of the superdifferential operators with matrix-valued coefficients are already contained in Refs.[7,12]. Hence, the generalization is quite straightforward and we shall be very brief.

We consider the superdifferential operators defined on (1|1) superspace with coordinates $X = (x, \theta)$:

$$L = D^m + U_1 D^{m-1} + \ldots + U_m \tag{4}$$

where $D = \partial_\theta + \theta\partial_x$ and U_i's are $n \times n$ matrices whose entries are N=1 superfields. It is always assumed that the operators are homogeneous under Z_2 grading; that is, $|U_i| = i(mod 2)$. For later uses we need to recall some basic definitions. First, we denote the Berezin integral by $\int_B = \int dx d\theta$ with the convention $\int d\theta\theta = 1$. The super Leibnitz rule for the covariant derivative D is

$$D^k \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} k \\ k-i \end{bmatrix} (-1)^{|\Phi|(k-i)} \Phi^{[i]} D^{k-i} \tag{5}$$

where k is an integer and $\Phi^{[i]} = (D^i\Phi)$ [*Note:* We also use the notations: $\Phi' = \Phi^{[1]}$ and $\Phi'' = \Phi^{[2]}$, etc. If $f(x)$ is an ordinary function, then $f'(x)$ means $\partial_x f(x)$], and the superbinomial coefficients $\begin{bmatrix} k \\ k-i \end{bmatrix}$ are defined by

$$\begin{bmatrix} k \\ k-i \end{bmatrix} = \left\{ \begin{array}{ll} 0, \quad for \quad i<0 \quad or \quad (k,k-i)\equiv(0,1) \quad (mod \quad 2) \\ \begin{pmatrix} [\frac{k}{2}] \\ [\frac{k-i}{2}] \end{pmatrix}, \quad otherwise, \end{array} \right\} \tag{6}$$

where $\begin{pmatrix} p \\ q \end{pmatrix}$ is the ordinary binomial coefficient. Given a superpseudodifferential operator $P = \sum p_i D^i$, where p_i's are $n\times n$ matrices, we define its super-residue as $sresP = p_{-1}$. It can be shown easily

$$\int_B tr \quad sres[P,Q] = 0 \tag{7}$$

where tr is the ordinary trace of matrix and $[P,Q] \equiv PQ - (-1)^{|P||Q|}QP$ is the supercommutator.

Given a functional $F[U] = \int_B f(U)$, where $f(U)$ is a Z_2-homogeneous differential polynomial of U_i's, we define its matrix-valued gradient X_F by

$$X_F = \sum_{i=1}^{m} (-1)^i D^{-m+i-1} \frac{\delta f}{\delta U_k} \tag{8}$$

where the matrix $\frac{\delta f}{\delta U_k}$ is defined by

$$\left(\frac{\delta f}{\delta U_k}\right)_{ij} = \sum_{l=0}^{\infty} (-1)^{|U_k|l + l(l+1)/2} D^l \frac{\partial f}{\partial (U_k)^{[l]}_{ji}} \tag{9}$$

With these definitions we now define the second Gelfand-Dickey bracket as

$$\{F,G\} = (-1)^{|F|+|G|+m} \int_B tr \quad sres[L(X_F L)_+ X_G - (LX_F)_+ LX_G] \tag{10}$$

where $(\quad)_+$ denotes the differetial part of a superpseudodifferential operator. The antisupersymmetry of the bracket is obvious by the virtue of the construction. The super Jacobi identity will follow from the Miura transformation which we shall introduce now.

Let us consider m $n \times n$ matrices Φ_i's whose entries are grassmanian odd N=1 superfields. We introduce a Poisson bracket defined by

$$\begin{aligned}\{F, G\}^* &\equiv \sum_{i=1}^{m}(-1)^{m+i+1}\int_B tr\left\{\left(\frac{\delta f}{\delta\Phi_i}\right)'\frac{\delta g}{\delta\Phi_i} - \Phi_i\left[\frac{\delta f}{\delta\Phi_i}, \frac{\delta g}{\delta\Phi_i}\right]\right\} \\ &= \sum_{i=1}^{m}(-1)^{m+i+1}\int_B tr\left\{\left[\nabla_i, \frac{\delta f}{\delta\Phi_i}\right]\frac{\delta g}{\delta\Phi_i}\right\}\end{aligned} \tag{11}$$

where $F[\Phi] = \int_B f(\Phi)$, $G[\Phi] = \int_B g(\Phi)$ and $\nabla_i \equiv D - \Phi_i$. One can show easily that the bracket (11) is indeed antisupersymmetric and satisfies the super Jacobi identity. As in the scalar case the two brackets (10) and (11) are actually equivalent once we make the following identification

$$\begin{aligned}L &= D^m + U_1 D^{m-1} + \ldots + U_m \\ &= (D - \Phi_1)(D - \Phi_2)\ldots(D - \Phi_m)\end{aligned} \tag{12}$$

Eq.(12) is the desired matrix Miura transformation. With the use of (7) the proof for this equivalenc is almost a repetition of that for the scalar case. We shall not spell it out here.

Since our main purpose is to get a supersymmetric version of V algebra, we have to consider the constraint $U_1 = 0$. Let us first discuss the constraint in the context of the bracket (11). From the matrix Miura transformation (12) we have

$$U_1 = \sum_{i=1}^{m}(-1)^i\Phi_i \tag{13}$$

To see whether this constraint is of second class or not we consider two functionals F and G which depend on Φ_i's only through U_1; that is, f and g are functions of U_1 only. By (13) we get $\frac{\delta f}{\delta\Phi_i} = (-1)^i\frac{\delta f}{\delta U_1}$ and a similar expression for g. Hence,

$$\{F, G\}^*|_{U_1=0} = \sum_{i=1}^{m}(-1)^{m+i+1}\int_B tr\left\{\left(\frac{\delta f}{\delta U_1}\right)'\frac{\delta g}{\delta U_1}\right\}_{U_1=0} \tag{14}$$

Clearly, the constraint is of second class only when m is odd. In this case the constraint imposed on $\frac{\delta f}{\delta\Phi_i}$'s is

$$\sum_{i=1}^{m}\left[\nabla_i, \frac{\delta f}{\delta\Phi_i}\right]_{U_1=0} = 0 \tag{15}$$

Following the procedure used in Refs.[7,12] we can trasnslate (15) into a constraint on X_F:

$$sres[L, X_F]|_{U_1=0} = 0 \tag{16}$$

which is more useful for computing brackets.

From now on we shall concern only with superdifferential operators of odd order

$$L = D^{2m+1} + U_2 D^{2m-1} + \ldots + U_{2m+1} \tag{17}$$

We note that if F and G depend linearly on U_i's through the traces then X_F and X_G are both proportional to the $n \times n$ idnetity matrix and hence the computations of the brackets are similar to the scalar case. In particular, if we define $J = tr(U_2)$ and $T = tr(U_3) - \frac{1}{2}tr(U_2')$ we find

$$\begin{aligned}
\{T(X), T(Y)\} &= [\frac{1}{4}nm(m+1)D^5 + \frac{3}{2}T(X)D^2 + \frac{1}{2}T'(X) + T''(X)]\delta(X-Y) \\
\{J(X), T(Y)\} &= [J(X)D^2 - \frac{1}{2}J'(X)D + J''(X)]\delta(X-Y) \\
\{J(X), J(Y)\} &= -[nm(m+1)D^3 + 2T(X)]\delta(X-Y)
\end{aligned} \tag{18}$$

where $\delta(X-Y) \equiv \delta(x-y)(\theta_X - \theta_Y)$. Eq.(18) is the classical N=2 super Virasoro algebra. We shall see later that the condition (16) leads to the presence of nonlocal terms in the resulting algebra. In other words, the second Gelfand-Dickey bracket associated with the operator (17) gives a nonlocal matrix generalization of classical N=2 W-superalgebra.

3. A Nonlocal Extension of Classical N=2 Super Virasoro Algebra

Now we shall work out in details the second Gelfand-Dickey bracket associated with the following operator

$$L = D^3 + U_2 D + U_3 \tag{19}$$

To this purpose it is convinient to evaluate first

$$J(X) \equiv L(XL)_+ - (LX)_+ L \tag{20}$$

where $X = D^{-3}X_1 + D^{-2}X_2 + D^{-1}X_3$. Staightforward algebras give

$$J(X) = J_1 D^2 + J_2 D + J_3$$

where

$$\begin{aligned}
J_1 &= (-1)^{|X|}\{-X_1' + X_2'' + X_3''' + [U_2, X_2] - [U_3, X_3] - (X_3U_2)'\} \\
J_2 &= X_1'' + X_2''' + [U_2, X_1] + U_2X_2' - (X_2U_2)' + U_3X_2 + U_3X_3' + (-1)^{|X|}X_2U_3 \\
J_3 &= (-1)^{|X|}\big\{ - X_1''' + X_3^{[5]} + U_2(X_3''' - X_1') - [U_3, X_1] + U_3X_3'' - (-1)^{|X|}(X_3U_3)'' \\
&\qquad - (X_3U_2)''' - (-1)^{|X|}(X_2U_3)' - U_2(X_3U_2)' - U_3X_3U_2 - (-1)^{|X|}U_2X_3U_3\big\}
\end{aligned} \tag{21}$$

Due to the constraint $U_1 = 0$ the coefficient of D^2, J_1, must vanish. This requirement, of course, is equivalent to the condition (16). We thus have $X_1 = X_2' + X_3'' - X_3U_2 + (D^{-1}[U_2, X_2]) - (D^{-1}[U_3, X_3])$. Putting this into the expressions of J_2 and J_3 we have

$$J_2 = J_{22} + J_{23} \qquad\qquad J_3 = J_{32} + J_{33}$$

where

$$\begin{aligned}
J_{22} &= 2X_2''' - (-1)^{|X|}X_2U_2 + [U_2', X_2] + 3[U_2, X_2'] + U_3X_2 \\
&\qquad + (-1)^{|X|}X_2U_3 + [U_2, (D^{-1}[U_2, X_2])] \\
J_{23} &= X_3^{[4]} + (X_3U_2)'' + U_3X_3' + [U_2, X_3''] - [U_3, X_3]' \\
&\qquad - [U_2, X_3U_2] - [U_2, (D^{-1}[U_3, X_3])] \\
J_{32} &= (-1)^{|X|}\big\{ - X_2^{[4]} - U_2X_2'' - (-1)^{|X|}(X_2U_3)' - [U_2, X_2]'' - [U_3, X_2'] \\
&\qquad - U_2[U_2, X_2] - [U_3, (D^{-1}[U_2, X_2])]\big\} \\
J_{33} &= (-1)^{|X|}\big\{U_3X_3'' - (-1)^{|X|}(X_3U_3)'' + [U_3'', X_3] + [U_2U_3, X_3] \\
&\qquad + [U_3, (D^{-1}[U_3, X_3])]\big\}
\end{aligned} \tag{22}$$

Using (8), (10) and (22) we can easily write down the desired brackets:

$$\begin{aligned}
&\{tr[fU_2(X)], tr[gU_2(Y)]\} \\
&\quad = tr\big\{ - 2fgD_X^3 - 2gf[U_3(x) - \frac{1}{2}U_2'(X)]\big\}\delta(X-Y) + tr\big\{[f, g]\big[3U_2(X)D_x \\
&\qquad - U_3(X) + 2U_2'(X)\big] + [U_2, f]D_X^{-1}[U_2, g]\big\}\delta(X-Y)
\end{aligned}$$

$$
\begin{aligned}
&\{tr[fU_2(X)], tr[gU_3(Y)]\} \\
&\quad = tr\big\{ - fgD_X^4 + gf\big[U_2(X)D_X^2 - U_3(X)D_X + U_2''(X)\big]\big\}\delta(X-Y) \\
&\qquad + tr\big\{[f,g]\big[2U_2(X)D_X^2 + U_3(X)D_X - U_3'(X) + U_2''(X)\big] - U_2(X)[U_2(X),f]g \\
&\qquad - [U_2(X),f]D_X^{-1}[U_2(X),g]\big\}\delta(X-Y) \qquad\qquad (23) \\
&\{tr[fU_3(X)], tr[gU_3(Y)]\} \\
&\quad = tr\big\{2gfU_3(X)D_X^2 + gfU_3''(X)\big\}\delta(X-Y) + tr\big\{[f,g]\big(U_3(X)D_X^2 \\
&\quad - U_2(X)U_3(X)\big) - [U_3(X),f]D_X^{-1}[U_3(X),g]\big\}\delta(X-Y)
\end{aligned}
$$

Here f and g are two constant real-valued matrices. We note that if either of f and g is proportional to the identity matrix then those terms involving at least a commutator would vanish and the resulting brackets resemble those in the scalar case. For example, if we define

$$
W_2(X) \equiv U_2(X) \qquad\qquad W_3(X) \equiv U_3(X) - \frac{1}{2}U_2'(X) \tag{24}
$$

then, similar to the scalar case, $tr(fW_2)$ and $tr(fW_3)$ form a N=2 supermultiplet, provided f is a traceless matrix. In other words, we have the following brackets:

$$
\begin{aligned}
\{tr[fW_2(X)], J(Y)\} &= -2tr[fW_3(X)]\delta(X-Y) \\
\{tr[fW_3(X)], J(Y)\} &= tr\big\{f[-W_2(X)D_X^2 + \frac{1}{2}W_2'(X)D_X - \frac{1}{2}W_2''(X)]\big\}\delta(X-Y) \\
\{tr[fW_2(X)], T(Y)\} &= tr\big\{f[W_2(X)D_X^2 - \frac{1}{2}W_2'(X)D_X + W_2''(X)]\big\}\delta(X-Y) \\
\{tr[fW_3(X)], T(Y)\} &= tr\big\{f[\frac{3}{2}W_3(X)D_X^2 + \frac{1}{2}W_3'(X)D_X + W''(X)]\big\}\delta(X-Y)
\end{aligned} \tag{25}
$$

We can also write down the other brackets. However, to be more specific, we shall focus on the case of 2×2 matrix. We parametrize the matrices W_2 and W_3 as follows

$$
W_2 = \begin{pmatrix} \frac{1}{2}J + \frac{1}{2}J_3 & J_+ \\ J_- & \frac{1}{2}J - \frac{1}{2}J_3 \end{pmatrix} \qquad W_3 = \begin{pmatrix} \frac{1}{2}T + \frac{1}{2}W_3 & W_+ \\ W_- & \frac{1}{2}T - \frac{1}{2}W_3 \end{pmatrix} \tag{26}
$$

Using (24) and the notation $\in (X-Y) \equiv D_X^{-1}\delta(X-Y)$ we obtain

$$
\begin{aligned}
\{J_\pm(X), J_\pm(Y)\} &= 2 \in (X-Y) J_\pm(X) J_\pm(Y) \\
\{J_\pm(X), J_\mp(Y)\} &= - \in (X-Y)[2J_\pm(X)J_\mp(Y) + J_3(X)J_3(Y)] \\
&\quad - [2D_X^3 + T(X)]\delta(X-Y) \mp [3J_3(X)D_X + \frac{3}{2}J_3'(X)]\delta(X-Y) \\
\{J_3(X), J_\pm(Y)\} &= 2 \in (X-Y) J_\pm(X) J_3(Y) \mp [6J_\pm(X)D_X + 3J_\pm'(X)]\delta(X-Y) \\
\{J_3(X), J_3(Y)\} &= -4 \in (X-Y)[J_+(X)J_-(Y) + J_-(X)J_+(Y)] \\
&\quad - 2[2D_X^3 + T(X)]\delta(X-Y) \\
\{J_\pm(X), W_\pm(Y)\} &= 2 \in (X-Y) J_\pm(X) W_\pm(Y) \\
\{J_\pm(X), W_\mp(Y)\} &= - \in (X-Y)[2J_\pm(X)W_\mp(Y) + J_3(X)W_3(Y)] \\
&\quad + \frac{1}{2}[J(X)D_X^2 - \frac{1}{2}J'(X)D_X + J''(X)]\delta(X-Y) \\
&\quad \mp [\frac{3}{2}W_3(X)D_X - W_3'(X) - \frac{1}{2}J(X)J_3(X)]\delta(X-Y) \\
\{J_\pm(X), W_3(Y)\} &= 2 \in (X-Y) J_3(X) W_\pm(Y) \\
&\quad \pm [3W_\pm(X)D_X - 2W_\pm'(X) - J(X)J_\pm(X)]\delta(X-Y) \\
\{J_3(X), W_\pm(Y)\} &= 2 \in (X-Y) J_\pm(X) W_3(Y) \\
&\quad \mp [3W_\pm(X)D_X - 2W_\pm'(X) - J(X)J_\pm(X)]\delta(X-Y) \\
\{J_3(X), W_3(Y)\} &= -4 \in (X-Y)[J_+(X)W_-(Y) + J_-(X)W_+(Y)] + \\
&\quad + [J(X)D_X^2 - \frac{1}{2}J'(X)D_X + J''(X)]\delta(X-Y) \\
\{W_\pm(X), W_\pm(Y)\} &= 2 \in (X-Y) W_\pm(X) W_\pm(Y) - \frac{1}{2}J_\pm(X)D_X J_\pm(X)\delta(X-Y) \\
\{W_\pm(X), W_\mp(Y)\} &= - \in (X-Y)[2W_\pm(X)W_\mp(Y) + W_3(X)W_3(Y)] \\
&\quad + \frac{1}{2}[D_X^5 + \frac{3}{2}T(X)D_X^2 + \frac{1}{2}T'(X)D_X + T''(X)]\delta(X-Y) \\
&\quad [\frac{1}{2}J_\pm(X)D_X J_\mp(X) + \frac{1}{4}J_3(X)D_X J_3(X)]\delta(X-Y) \\
&\quad \pm \frac{1}{2}[J(X)W_3(X) + J_3(X)T(X)]\delta(X-Y) \\
&\quad \pm [\frac{3}{4}J_3(X)D_X^3 + \frac{3}{8}J_3'(X)D_X^2 + \frac{3}{8}J_3''(X)D_X]\delta(X-Y)
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\{W_3(X), W_\pm(Y)\} = 2 \in (X-Y) W_\pm(X) W_3(Y) - \frac{1}{2} J_\pm(X) D_X J_3(X) \delta(X-Y) \\
\pm [J(X) W_\pm(X) + J_\pm(X) T(X)] \delta(X-Y) \\
\pm [\frac{3}{2} J_\pm(X) D_X^3 + \frac{3}{4} J'_\pm(X) D_X^2 + \frac{3}{4} J''_\pm(X) D_X] \delta(X-Y) \\
\{W_3(X), W_3(Y)\} = -4 \in (X-Y)[W_+(X) W_-(Y) + W_-(X) W_+(Y)] \\
+ [D_X^5 + \frac{3}{2} T(X) D_X^2 + \frac{1}{2} T'(X) D_X + T''(X)] \delta(X-Y) \\
+ [J_+(X) D_X J_-(X) + J_-(X) D_X J_+(X)] \delta(X-Y)
\end{aligned}
$$

We like to remark here that the superalgebra given by (27) is invariant under the transformations:

$$
\begin{aligned}
J_\pm \longrightarrow J_\mp \qquad J_3 \longrightarrow -J_3 \\
W_\pm \longrightarrow W_\mp \qquad W_3 \longrightarrow -W_3
\end{aligned}
\tag{28}
$$

The transformations (28) is due to the invariance of the second Gelfand-Dickey bracket under the constant similarity transformation:

$$
L \longrightarrow S^{-1} L S \qquad X_{F(G)} \longrightarrow S^{-1} X_{F(G)} S \tag{29}
$$

Here S is a constant nonsingular matrix. Indeed, if we take $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then the first of (29) gives (28).

The other remark is that the Dirac bracket arising from setting J and $J_a (a = 3, \pm)$ to zero would involve correction terms proportional to $[2D^3 + T]^{-1}$. This makes these brackets to have infinite number of nonlocal terms. Hence, this reduction is not well defined. Therefore how to get a nonlocal matrix generalization of N=1 Super Virasoro algebra remains an interesting question.

4. $V_{2,2}$ algebra As Dirac Bracket

In the end of the last section we see that eliminating the spin-1 superfields does not give a well defined N=1 superalgebra. However, eliminating the fermionic fields in a superalgebra is quite straightforward. To get the bosonic limit of the N=2 superalgebra

defined by (18), (25), (26) and (27) we write the superfields in component form:

$$
\begin{aligned}
T(X) &= \phi(x) - \theta_X t(x) \qquad J(X) = j(x) + \theta_X \varphi(x) \\
W_a(X) &= \phi_a(x) + \theta_X v_a(x) \qquad J_a(X) = j_a(x) + \theta_X \varphi_a(x)
\end{aligned}
\tag{30}
$$

Setting ϕ and φ to be zero in (18) (with n=2 and m=1) we have

$$
\begin{aligned}
\{t(x), t(y)\} &= [-\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y) \\
\{j(x), t(y)\} &= [j(x)\partial_x + j'(x)]\delta(x-y) \\
\{j(x), j(y)\} &= -4\partial_x \delta(x-y)
\end{aligned}
\tag{31}
$$

which is, as expected, the statement that t is the Virasoro generator and j is a spin-1 field. While (25) gives (for $a = \pm, 3$)

$$
\begin{aligned}
\{j_a(x), j(y)\} &= 0 \\
\{v_a(x), j(y)\} &= [-j_a(x)\partial_x]\delta(x-y) \\
\{j_a(x), t(y)\} &= [j_a(x)\partial_x + j_a'(x)]\delta(x-y) \\
\{v_a(x), t(y)\} &= [2v_a(x)\partial_x + v_a'(x)]\delta(x-y)
\end{aligned}
\tag{32}
$$

Finally, the bosonic limit of (27) are

$$
\begin{aligned}
\{j_\pm(x), j_\pm(y)\} &= 2\epsilon(x-y) j_\pm(x) j_\pm(y) \\
\{j_\pm(x), j_\mp(y)\} &= -\epsilon(x-y)[2j_\pm(x)j_\mp(y) + j_3(x)j_3(y)]\delta(x-y) \\
&\quad - 2\delta'(x-y) \mp 3j_3(x)\delta(x-y) \\
\{j_3(x), j_\pm(y)\} &= 2\epsilon(x-y) j_\pm(x) j_3(y) \mp 6 j_\pm(x)\delta(x-y) \\
\{j_3(x), j_3(y)\} &= -4\epsilon(x-y)[j_+(x)j_-(y) + j_-(x)j_+(y)] - 4\delta'(x-y) \\
\{j_\pm(x), v_\pm(y)\} &= 2\epsilon(x-y) j_\pm(x) v_\pm(y) \\
\{j_\pm(x), v_\mp(y)\} &= -\epsilon(x-y)[2j_\pm(x)v_\mp(y) + j_3(x)v_3(y)] - \frac{1}{2}[j(x)\partial_x + j'(x)]\delta(x-y) \\
&\quad \mp [v_3(x) + \frac{1}{2}j(x)j_3(x)]\delta(x-y) \\
\{j_\pm(x), v_3(y)\} &= 2\epsilon(x-y) j_3(x) v_\pm(y) \pm [2v_\pm(x) + j(x)j_\pm(x)]\delta(x-y) \\
\{j_3(x), v_\pm(y)\} &= 2\epsilon(x-y) j_\pm(x) v_3(y) \mp [2v_\pm(x) + j(x)j_\pm(x)]\delta(x-y) \\
\{j_3(x), v_3(y)\} &= -4\epsilon(x-y)[j_+(x)v_-(y) + j_-(x)v_+(y)] - [j(x)\partial_x + j'(x)]\delta(x-y)
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\{v_\pm(x), v_\pm(y)\} &= 2\epsilon(x-y)v_\pm(x)v_\pm(y) + \frac{1}{2}j_\pm(x)\partial_x j_\pm(x)\delta(x-y)\\
\{v_\pm(x), v_\mp(y)\} &= -\epsilon(x-y)[2v_{\pm(x)}v_\mp(y) + v_3(x)v_3(y)] + \frac{1}{2}[-\partial_x^3 + 2t(x)\partial_x\\
&\quad + t'(x)]\delta(x-y) - [\frac{1}{2}j_\pm(x)\partial_x j_\mp(x) + \frac{1}{4}j_3(x)\partial_x j_3(x)]\delta(x-y)\\
&\quad \mp [\frac{3}{4}j_3(x)\partial_x^2 + \frac{3}{4}j_3'(x)\partial_x + \frac{1}{2}j(x)v_3(x) + \frac{1}{2}j_3(x)t(x)]\delta(x-y)\\
\{v_3(x), v_\pm(y)\} &= 2\epsilon(x-y)v_\pm(x)v_3(y) + \frac{1}{2}j_\pm(x)\partial_x j_3(x)\delta(x-y)\\
&\quad \mp [\frac{3}{2}j_\pm(x)\partial_x^2 + \frac{3}{2}j_\pm'(x)\partial_x + j(x)v_\pm(x) + j_\pm(x)t(x)]\delta(x-y)\\
\{v_3(x), v_3(y)\} &= -4\epsilon(x-y)[v_+(x)v_-(y) + v_-(x)v_+(y)] + [-\partial_x^3\\
&\quad + 2t(x)\partial_x + t'(x)]\delta(x-y) - [j_+(x)\partial_x j_-(x) + j_-(x)\partial_x j_+(x)]\delta(x-y)
\end{aligned}
$$

This algebra contains, besides the Virasoro generator t, three spin-2 fields, $v_a(a=\pm,3)$ and four spin-1 fields, j and $j_a(a=\pm,3)$. The four spin-1 fields form a closed subalgebra. In fact, the local parts of the brackets $\{j_a(x), j_b(x)\}$ define nothing but the ordinary $sl(2,R)$ Kac- Moody algebra. Hence, the first four brackets of (33) give a example of nonlocal generalization of Kac-Moody algebra.

Since the $V_{2,2}$ algebra consists of a Virasoro generator and three spin-2 fields, it is natural to expect that it can be obtained from the above algebra by setting the four spin-1 fields to be zero. We now show that this expectation is actually true. From (31) we see that $\{j_a(x), j(y)\} = 0$ and $\{v_a(x), j(y)\}|_{j_a=0} = 0$. A little thinking tells us that to take care of the constraint $j=0$ we simply drop j-dependent terms in (33). Assuming that this has been done we note further that $\{v_a(x), j_b(y)\}|_{j_\pm, j_3=0}$ are not all zero. Some corrections of the brackets must be taken. Since the 3×3 matrix $\{j_a(x), j_b(y)\}|_{j_\pm, j_3=0}$ is formally invertible, we can simply use the formula of Dirac bracket:

$$
\begin{aligned}
&\{v_a(x), v_b(y)\}^{Dirac} = \{v_a(x), v_b(y)\}|_{j_\pm, j_3=0}\\
&\quad - \sum_{c,d=\pm,3}\int\int dwdz\{v_a(x), j_c(w)\}|_{j_\pm,j_3=0}M_{c,d}(w,z)\{j_d(z), v_b(y)\}|_{j_\pm,j_3=0}
\end{aligned}
\tag{34}
$$

where $M_{c,d}(w,z)$ is the inverse of the matrix $\big(\{j_c(w), j_d(z)\}|_{j_\pm,j_3=0}\big)$. The second piece is the correction to the original bracket due to the constraints $j_\pm, j_3 = 0$. In operator form

it reads

$$
-\begin{pmatrix} 0 & -v_3 & 2v_+ \\ v_3 & 0 & -2v_- \\ -2v_+ & 2v_- & 0 \end{pmatrix}
\begin{pmatrix} 0 & -\frac{1}{2}\partial^{-1} & 0 \\ -\frac{1}{2}\partial^{-1} & 0 & 0 \\ 0 & 0 & -\frac{1}{4}\partial^{-1} \end{pmatrix}
\begin{pmatrix} 0 & -v_3 & 2v_+ \\ v_3 & 0 & -2v_- \\ -2v_+ & 2v_- & 0 \end{pmatrix}
$$

$$
= -\begin{pmatrix} v_+\partial^{-1}v_+ & -\frac{1}{2}v_3\partial^{-1}v_3 - v_+\partial^{-1}v_- & v_3\partial^{-1}v_+ \\ -\frac{1}{2}v_3\partial^{-1}v_3 - v_-\partial^{-1}v_+ & v_-\partial^{-1}v_- & v_3\partial^{-1}v_- \\ v_+\partial^{-1}v_3 & v_-\partial^{-1}v_3 - 2v_-\partial^{-1}v_+ & -2v_+\partial^{-1}v_- \end{pmatrix} \tag{35}
$$

Each correction term is purely nonlocal and, in fact, equals to negative of one half of the nonlocal term in the corresponding uncorrected bracket. Therefore the resulting Dirac brackets read

$$
\begin{aligned}
\{v_\pm(x), v_\pm(y)\}^{Dirac} &= \epsilon(x-y)v_\pm(x)v_\pm(y) \\
\{v_\pm(x), v_\mp(y)\}^{Dirac} &= -\epsilon(x-y)[v_\pm(x)v_\mp(y) + \frac{1}{2}v_3(x)v_3(y)] \\
&\quad + \frac{1}{2}[-\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y) \\
\{v_3(x), v_\pm(y)\}^{Dirac} &= \epsilon(x-y)v_\pm(x)v_3(y) \\
\{v_3(x), v_3(y)\}^{Dirac} &= -2\epsilon(x-y)[v_+(x)v_-(y) + v_-(x)v_+(y)] \\
&\quad + [-\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x-y)
\end{aligned} \tag{36}
$$

which, together with the first bracket in (31) and the last bracket in (32), define the $V_{2,2}$ algebra. More precisely, it is the second Gelfand-Dickey bracket associated with $L = \partial^2 - U$, where U is parametrized by

$$
U = \begin{pmatrix} \frac{1}{2}t + \frac{1}{2}v_3 & v_+ \\ v_- & \frac{1}{2}t - \frac{1}{2}v_3 \end{pmatrix} \tag{37}
$$

Of course, $V_{2,2}$ algebra can be regarded as the Dirac bracket arising from the local algebra defined by the second Gelfand-Dickey bracket associated with $L = \partial^2 + V\partial - U$ by setting the 2×2 matrix V to zero. What we have shown here is that it can also arise from a nonlocal algebra.

This work was supported by the National Science Council of Republic of China under Grant number NSC-83-0208-M-007-008.

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